Diffgeo 1

Peter Smillie — I'm american, so you can call me Peter,
(or Prof. Smillie)

Logistics

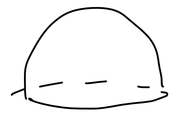
- HW weekly due Friday; not this week.
  (if ~100 people, sol'ns posted, if <<100 people, not?)
- If <50 people, option to skip W/F
- Written exam TBD, likely July
- Recording (maybe?)

Outline

I. Course outline

- (smooth) manifolds –

2-dim'l
just the surface —
infinitely thin



leg connection,
don't include
the equator.

- Riemann's POV — 1854 lecture / (pub. 1866) Göttingen
Über die Hypothesen,
welche der Geometrie
zu Grunde liegen.
(Spivak vol 2.)

trying to capture concept of non-discrete spaces
multiply-extended quantities

- color
- positions of objects } perhaps the only outside of higher math
- many-valued analytic functions $\int \frac{dz}{z}$, $\int \frac{dz}{\sqrt{(z-a)(z-b)(z-c)}}$

<u>Other notes</u>: 2 inductive ways of understanding — fns in + fns out.

- space

- Riemannian manifolds
  - same picture but different notion of "the same"
  - bending w/out stretching
  - open problem: is every closed surface in $\mathbb{R}^3$ rigid?

$\boxed{\text{Curvature}}$

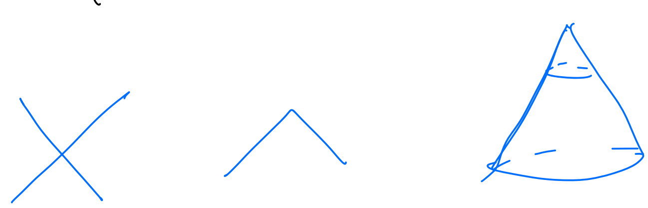
  - general program: to what extent does the (infinitely) small determine the (infinitely) large?
(Riemann's example: spherical universe.)

II. When is a subset of $\mathbb{R}^n$ a d-dimensional manifold?

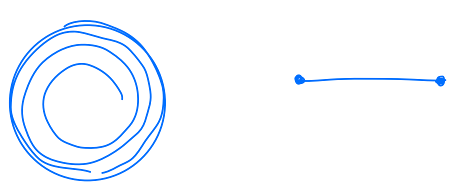
<u>1827</u> Disquisitiones generales circa superficies curvas
(Spivak vol 2)

· <u>Guess</u> A subset $S \subseteq \mathbb{R}^n$ is a d-dimensional mfld if it is infinitesimally a d-dim'l plane at each point.

- torus example
- non-examples:



questionable:
(rather not)



1-d : curve
2-d : surface

Aside: smooth functions:

If $f(x_1,\dots,x_n):\mathbb{R}^n\longrightarrow\mathbb{R}$, the partial deriv wrt x_i is

$$\frac{\partial f}{\partial x_i}(x_1,\dots x_n) := \lim_{\varepsilon\to 0}\frac{f(x_1\dots x_i+\varepsilon,\dots x_n)-f(x_1\dots x_n)}{\varepsilon}$$

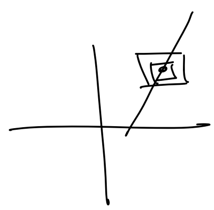
<u>Defn</u>. A fn. $f(x_1,\dots,x_n):\mathbb{R}^n\longrightarrow\mathbb{R}$ is <u>smooth</u>, or C^∞, if all partial derivs $\frac{\partial^{(k)} f}{\partial x_{i_1}\dots\partial x_{i_k}}$ exist and are continuous.

- $f:\mathbb{R}^n\to\mathbb{R}^m$ C^∞ if each component is

If f is smooth, $p\in\mathbb{R}^n$, the derivative df_p is the linear map $\mathbb{R}^n\to\mathbb{R}^m$ represented by the matrix $\frac{\partial f^i}{\partial x_j}$.

$$f(p+x) = f(p) + df_p x + o(x)$$

$\Longrightarrow$ the graph of a smooth fn should be manifold



Formalization of Gauss's condition:

<u>Recall</u> A set $U\subseteq\mathbb{R}^n$ is <u>open</u> if $x\in U\Rightarrow B_\varepsilon(x)\in U$ for some ε.

- A <u>neighborhood</u> of x is an open set containing x
- A property $P:\{\text{subsets of }\mathbb{R}^n\}\to\{\text{true},\text{false}\}$ is <u>locally</u> true on S if $\forall x\in S$, $\exists$ nbhd U of x s.t. $P(U\cap S)$

<u>Thus</u> f is smooth iff it is locally smooth $\left(P: \text{top}/\mathbb{R}\to\{t,f\}\right)$

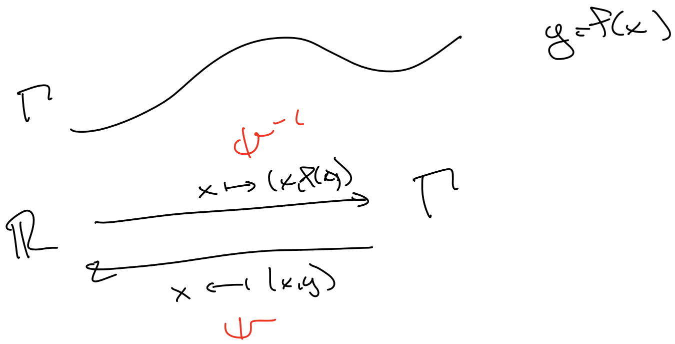
<u>Defn</u>. If S is a subset of $\mathbb{R}^n$, we say $f:S\longrightarrow\mathbb{R}$ is smooth if it is locally the restriction of some smooth $F:U\longrightarrow\mathbb{R}$ to $U\cap S$.

- $\rightsquigarrow f: S^{\in\mathbb{R}^n}\to T^{\in\mathbb{R}^m}$ smooth

Dfn $f : S \longrightarrow T$ (smooth) is a diffeomorphism if f^{-1} is also smooth.

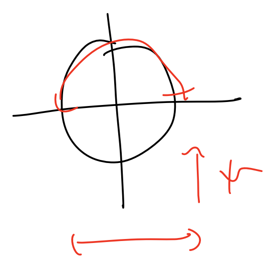
Dfn $S \subseteq \mathbb{R}^n$ is a d-dim'l manifold if S is locally diffeomorphic to an open subset of $\mathbb{R}^d$

Check that the graph of a smooth fn is a mfld

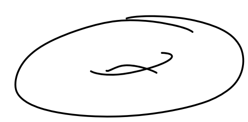


Cor: If $S \subseteq \mathbb{R}^n$ is locally the graph of a smooth fn from some d coordinates to the other $n-d$, then it is a d-mfld.

eg S^1



eg



Dfn The maps ψ are called charts. The fns

$$x_i \circ \psi^{-1}$$

are called local coordinates.

Note If ψ_1, ψ_2 are 2 charts containing a point q, then $\psi_1^{-1} \circ \psi_2$ is smooth (where defined) (a nontrivial)

[comp of smooth is smooth (easy ex)]

Some trivial cases:

d-dim'l submfld of $\mathbb{R}^d$?

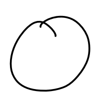
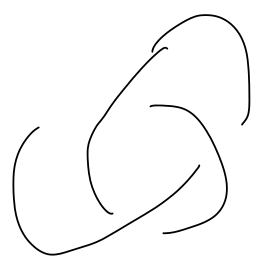
0-dim'l submfld of $\mathbb{R}^n$?

(Hint: a discrete subset of $\mathbb{R}^n$ is countable;
pf. choose a rat'l point in each open)

<u>Diffeomorphism</u>: <u>Diff'l topology</u> / <u>manifold theory</u> (1st part of this course) is the study of properties invariant under <u>diffeomorphism</u>.

<u>Example</u>

• isotopy of $\mathbb{R}^3$

•  $\subset \mathbb{R}^2$ $\simeq$  $\subset \mathbb{R}^3$

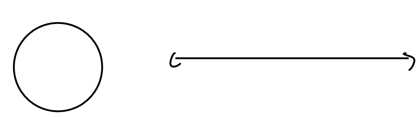
(ex?) • A circle and $\mathbb{R}$ are <u>not</u> diffeomorphic

Note diffeomorphism does not seem to care much about the ambient space.

Often, we encounter things that we feel must be mflds, but don't come with a particular embedding.

<u>Abstract manifolds</u>

<u>baby example</u>: $S^1 \sqcup$ x-axis



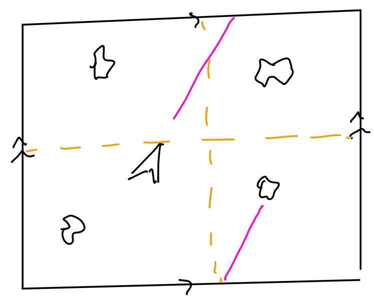
(local property → $\sqcup$ of mflds should be one)

they intersect! Must push them apart. Easy to do, but not canonically!

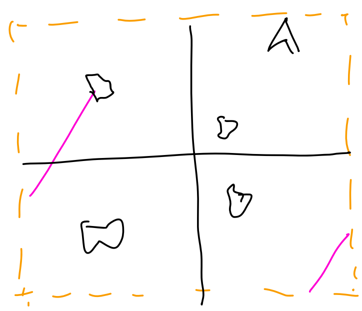
(Abstract) manifolds

Ex:

Asteroids (Atari, 1979)



This also has natural smooth charts to $\mathbb{R}^2$



So we can still describe smooth motion on it, so we should think of it as a manifold. Is it?

- A topological space is a set M, plus a notion of open subsets, s.t. finite $\cap$ + arb. $\cup$ is open. (We almost always just call it S)
- A subset of a top space aquires a topology by 'intersection,

Standard defn

- A d-dim'l manifold can be presented by:
  1. A Hausdorff topological space M with a countable basis.
  2. An atlas of charts:

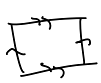
     A collection of open set $U_\alpha \subseteq \mathbb{R}^d$ and continuous maps
     $$\varphi_\alpha : U_\alpha \longrightarrow M$$
     such that
     - the sets $\varphi_\alpha(U_\alpha)$ cover M
     - each φ_α is a homeomorphism onto its image
     - $\forall\, \alpha$ and β, $\varphi_\alpha^{-1} \circ \varphi_\beta$ is smooth where its defined
- Atlases $\{\varphi_\alpha\}$ and $\{\psi_\alpha\}$ are equivalent if $\forall\, \alpha$ and β, $\varphi_\alpha \circ \psi_\beta^{-1}$ is smooth where defined.

Topological terms (culture — we'll check these with a thm. not by hand, if we need to)

- S is Hausdorff if every pair of points has disjoint nbhds
- A basis of S is a collection of open sets that generate all opens under $\cup$ (e.g. $B_\varepsilon(p)$ for $\varepsilon \in \mathbb{Q}^+$, $p \in \mathbb{Q}^n$).

Check ① A submfld of $\mathbb{R}^n$ is a manifold

Haus ✓, 2nd count ✓, charts ✓

②  defines a mfld.

Ended here

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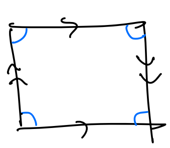
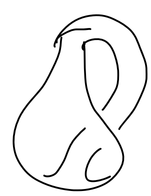
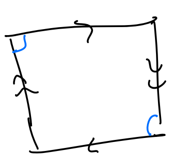
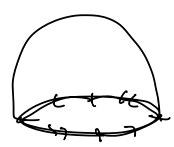
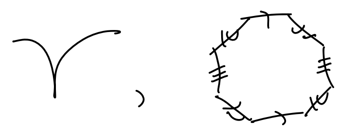
③  ✓ 

 × but 

Fact Every abstract mfld is diffeomorphic to a submfld of $\mathbb{R}^n$ for large enough n (in fact, $n = 2d$; $2d+1$ is pretty easy)

Culture If you replace diffeo w/ homeo, you get a topological mfld

(ish) • Then all atlases on M are equivalent — it's a property of the underlying space.

• includes ex's like ;

• Exotic spheres

Diffeomorphism